

Relativistic Orbits and the Zeros of $\wp(\Theta)$

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Abstract

A simple expression for the zeros of Weierstrass' function is given which follows from a formula for relativistic orbits.

Index Terms — Elliptic functions, Weierstrass' function, Relativistic Orbits.

1 Introduction

For elliptic functions, Weierstrass' function $\wp(\Theta)$ is a very important tool. It appears in countless applications in mathematics, physics, and engineering. There is some interest to determine its zeros. This problem is considered as being difficult. A formula can be found in [1], see also [6]. However, to quote ([6], p.105): "There seems to be no simple way to express the roots".

In this contribution a simple expression is given for the roots of $\wp(\Theta) = 0$. The formula follows from a formula for relativistic orbits (see [2] and [4]), which in turn originated from research in the field of Cauer filters [3]. We assume that the reader is familiar with elliptic function theory. We use the notation employed by Lawden [5].

2 Relativistic Orbits

In Lawden's book ([5], pp.126-129), it has been shown, that the generalisation of the Kepler ellipse to the relativistic orbit leads to the polar equation

$$r = \frac{1}{A + B \cdot sn^2(C\Theta, k)} ,$$

where r gives the distance of two bodies and Θ the angle in the plane of motion with suitable reference. As Lawden was mainly interested in deriving the perihel shift of the planet Mercury, he only determined approximations of the constants A , B , and C .

In [2] and [3] these constants have been determined exactly by explicitly solving the orbit differential equation for $u = 1/r$ given by

$$u'' + u = \alpha + 3\beta \cdot u^2 .$$

The dash denotes derivation with respect to the angle Θ . The constants α and β depend on the motion to be considered (e.g. planetary motion, photonic motion, motion of elementary particles, etc.). This led to

$$\begin{aligned} A &= \frac{1}{6\beta} \cdot \left(1 - \sqrt{\frac{1-12\alpha\beta}{k^4-k^2+1}}(k^2+1) \right) \\ B &= \frac{k^2}{2\beta} \cdot \sqrt{\frac{1-12\alpha\beta}{k^4-k^2+1}} \\ C &= \frac{1}{2} \cdot \sqrt[4]{\frac{1-12\alpha\beta}{k^4-k^2+1}} . \end{aligned} \quad (1)$$

Note that the constant C is invariant to the transformation $k^2 \rightarrow k'^2 = 1 - k^2$.

The shape for the relativistic motion (and for the magnitude of Cauefilters) has two relevant constants, the value of k and the value of the product $\alpha\beta$. These two parameters essentially determine the trajectories (up to scaling).

3 Simple Expression for Zeros of \wp

The orbit equation can be easily solved for the angle $\Theta = (1/C) \cdot \text{sn}^{-1}(\sqrt{\frac{1/r-A}{B}})$. We now show the relation of the orbit equation to the Weierstrass function \wp . To this end we use the relation of \wp to the sn^2 -function (see [7], p. 553)

$$\wp(\Theta) = \frac{e_1 - e_3}{\text{sn}^2(\sqrt{e_1 - e_3}\Theta)} + e_3 ,$$

or

$$\wp(\Theta) = \frac{C^2}{\text{sn}^2(C\Theta)} - \frac{(1+k^2) \cdot C^2}{3} , \quad (2)$$

i.e. $C \stackrel{!}{=} \sqrt{e_1 - e_3}$ and $e_3 = -\frac{1+k^2}{3}C^2$. The value C is given by equation (1). The angle Θ then can be expressed by the integral

$$\Theta = \int_{\infty}^{\wp(\Theta)} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} ,$$

where g_2 and g_3 are the Weierstrass invariants which follow from the differential equation

$$\wp'^2 = 4\wp^3 - g_2 \wp - g_3$$

and can be expressed as sum over all non-zero reciprocal 4-th and 6-th powers respectively of the lattice points of the adjoint parallelograms. The polynomial $4y^3 - g_2 y - g_3$ factors as

$$4y^3 - g_2 y - g_3 = 4(y - e_1)(y - e_2)(y - e_3) .$$

Thus with $C = \sqrt{e_1 - e_3}$, $e_3 = -\frac{1+k^2}{3}C^2$, and $e_1 + e_2 + e_3 = 0$ we get

$$\begin{aligned} e_1 &= -\frac{k^2-2}{3}C^2 \\ e_2 &= \frac{2k^2-1}{3}C^2 \\ e_3 &= -\frac{k^2+1}{3}C^2 , \end{aligned}$$

which leads to

$$\begin{aligned} g_3 &= 4e_1e_2e_3 \Rightarrow g_3 = \frac{4}{27}(k^2+1)(k^2-2)(2k^2-1)C^6 \\ g_2 &= -4(e_1e_2+e_1e_3+e_2e_3) \Rightarrow g_2 = \frac{4}{3}(k^4-k^2+1)C^4. \end{aligned}$$

Using equation (1) we obtain the remarkably simple result

$$g_2 = \frac{1}{12} - \alpha\beta, \quad (3)$$

and

$$g_3 = \frac{(k^2+1)(k^2-2)(2k^2-1)}{2^4 \cdot 3^3} \left(\frac{1-12\alpha\beta}{k^4-k^2+1} \right)^{3/2}.$$

To obtain a compact description of all cases we compute the discriminant of the polynomial $4y^3 - g_2y - g_3$ which is given by $g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2$, thus $g_2^3 - 27g_3^2 = 2^4(1 - k^2)^2k^4C^{12}$ which leads to

$$g_2^3 - 27g_3^2 = \frac{(1 - k^2)^2 \cdot k^4 \cdot (1 - 12\alpha\beta)^3}{2^8 \cdot (k^4 - k^2 + 1)^3}.$$

If the discriminant is greater than zero we have three real roots, if it is zero we have three real roots which are not all distinct and for negative discriminant we have one real and two complex zeros. The absolute invariant $g_2^3/(27g_3^2)$ follows as

$$\frac{g_2^3}{27g_3^2} = \frac{(k^4 - k^2 + 1)^3}{(k^2 + 1)^2 \cdot (k^2 - 2)^2 \cdot (k^2 - \frac{1}{2})^2}. \quad (4)$$

The absolute invariant transforms the three cases of the discriminant to the cases that $g_2^3/(27g_3^2)$ is greater, equal or smaller than 1. The absolute invariant is an extremely useful tool for characterizing the ratio of the two periods of the corresponding elliptic function. It is immediately seen that the absolute invariant is left unchanged by the transformations $k^2 \rightarrow k'^2 = 1 - k^2$ (like C) and $k^2 \rightarrow 1/k^2$.

The zeros of \wp now follow from equation (2)

$$\frac{C^2}{sn^2(C\Theta)} = \frac{(1 + k^2) \cdot C^2}{3} \Rightarrow \Theta_0 = \pm \frac{1}{C} \cdot sn^{-1}\left(\sqrt{\frac{3}{1 + k^2}}, k\right).$$

hence the zeros of the Weierstrass function are given by the simple expression

$$\Theta_0 = \pm \sqrt[4]{\frac{4(k^4 - k^2 + 1)}{3g_2}} \cdot sn^{-1}\left(\sqrt{\frac{3}{1 + k^2}}, k\right). \quad (5)$$

We give some examples.

Example 1 Computing the length of the famous Lemniscate, one encounters the integral $\int \frac{dx}{\sqrt{1-x^4}}$ which can be transformed to the integral $\int \frac{dy}{\sqrt{4y^3-4y}}$. Hence $g_2 = 4$ and $g_3 = 0$. The values of k^2 follow from equation (4). The possible values are $k^2 = -1$, $k^2 = 1/2$, and $k^2 = 2$. We select the value of k^2 in the interval $[0, 1]$, i.e. $k^2 = 1/2$ and obtain

$$\Theta_0 = \pm \frac{1}{\sqrt{2}} sn^{-1}\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right).$$

The approximate numeric value in the parallelogram around zero is given by $\approx \pm 1.3110287771 \cdot (1 - i)$. Clearly this is the value $\pm \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}} \cdot (1 - i)$. Note that for $k = i$ we have Gauss' Sinus Lemniscatus $sl(z) = sn(z, i) = \frac{1}{2} sd(\sqrt{2}z, \frac{1}{\sqrt{2}})$.

In table I data for some values of k used in the following examples are collected. All numeric values for Θ_0 given in the examples are for the parallelogram around zero.

Example 2 Let $g_2 = 7$, $g_3 = 3$, and $k = \frac{1}{\sqrt{5}}$ (see table I). This leads to

$$\Theta_0 = \pm \sqrt{\frac{2}{5}} \cdot sn^{-1}\left(\sqrt{\frac{5}{2}}, \frac{1}{\sqrt{5}}\right) \approx \pm(1.0496381 - i \cdot 0.77781243) .$$

Example 3 Let $g_2 = 11$, $g_3 = 7$, and $k = \frac{\sqrt{2}-1}{\sqrt{2}+1}$. This leads to

$$\Theta_0 = \pm \sqrt{2} \sqrt[4]{17 - 12\sqrt{2}} \cdot sn^{-1}\left(\sqrt{\frac{3}{2} + \sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \approx \pm(0.9270373 + i \cdot 0.6766441) .$$

Example 4 Let $g_2 = 15$, $g_3 = \sqrt{2} \cdot 7$, and $k = \sqrt{2} - 1$. This leads to

$$\Theta_0 = \pm \sqrt{\frac{2}{3}} \sqrt[4]{3 - 2\sqrt{2}} \cdot sn^{-1}\left(\frac{\sqrt{3}}{2} \sqrt{2 + \sqrt{2}}, \sqrt{2} - 1\right) \approx \pm(0.86473386 + i \cdot 0.637892607) .$$

We may also use the property

$$\wp(\lambda\Theta | \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}) = \frac{\wp(\Theta | g_2, g_3)}{\lambda^2}$$

to obtain additional solutions from known values of g_2 and g_3 for any $\lambda \neq 0$.

Example 5 Using $\lambda = 2^{1/12}$, we get the zeros of the Weierstrass function for the case $g_2 = 15/2^{1/3}$ and $g_3 = 7$ from the previous example as

$$\Theta_0 = \pm 2^{1/12} \cdot \sqrt{\frac{2}{3}} \sqrt[4]{3 - 2\sqrt{2}} \cdot sn^{-1}\left(\frac{\sqrt{3}}{2} \sqrt{2 + \sqrt{2}}, \sqrt{2} - 1\right) .$$

In general for given g_2 and g_3 we can easily compute k from equation (4). Using the substitution $\xi = k^2 + 1/k^2 - 1$ we get a polynomial of degree three in ξ which can be solved for ξ from which k^2 and k follows. Namely we get the polynomial

$$\xi^3 + a \cdot \xi - a = 0$$

where ¹ $a = \frac{27}{4} \cdot \frac{g_2^3}{27g_3^2 - g_2^3}$, which solved for ξ yields

$$\xi = \rho \cdot \sqrt[3]{\frac{a}{2} + \sqrt{\left(\frac{a}{3}\right)^3 + \left(\frac{a}{2}\right)^2}} + \rho^2 \cdot \sqrt[3]{\frac{a}{2} - \sqrt{\left(\frac{a}{3}\right)^3 + \left(\frac{a}{2}\right)^2}}$$

with $\rho \in \{1, e^{i2\pi/3}, e^{i4\pi/3}\}$. Then all values of k^2 follow:

$$k^2 = \frac{1}{2} \cdot (\xi + 1 \pm \sqrt{\xi^2 + 2\xi - 3}) . \quad (6)$$

¹Note that a is simply related to the j-invariant $1728 g_2^3 / (g_2^3 - 27g_3^2)$.

If a real solution for k^2 in the interval $[0, 1]$ exists it is obtained by selecting $\rho = 1$ and the minus sign in equation (6).

Table I: Data for some invariants

k	$\frac{(k^4 - k^2 + 1)^3}{(k^2 + 1)^2 (k^2 - 2)^2 (k^2 - \frac{1}{2})^2}$	$\frac{g_2^3}{27 \cdot g_3^2}$	g_2	g_3
$\frac{1}{\sqrt{5}}$	$\frac{7^3}{3^5}$	$\frac{7^3}{27 \cdot 3^2}$	7	3
$\frac{\sqrt{2}-1}{\sqrt{2}+1}$	$\frac{11^3}{3^3 7^2}$	$\frac{11^3}{27 \cdot 7^2}$	11	7
$\sqrt{2} - 1$	$\frac{5^3}{2 \cdot 7^2}$	$\frac{(3 \cdot 5)^3}{27 \cdot (\sqrt{2} \cdot 7)^2}$	15	$\sqrt{2} \cdot 7$

4 Conclusion

The simple and easy to compute expression

$$\Theta_0 = \pm \sqrt[4]{\frac{4(k^4 - k^2 + 1)}{3 g_2}} \cdot \operatorname{sn}^{-1}\left(\sqrt{\frac{3}{1 + k^2}}, k\right)$$

for the zeros of the Weierstrass elliptic function \wp has been derived, where k is given in closed form as function of g_2 and g_3 .

References

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